

Deriving Special Relativity

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The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

Hermann Minkowski (1864–1909)

1 Introduction

Many people are familiar with the strange phenomena of special relativity—clocks slowing down, objects shrinking, twins aging at different rates, and so on—but regard their explanations as deep mysteries of higher mathematics. One of the many amazing things about special relativity—unlike general relativity—is that its basic predictions can be explained using simple algebra. Even the basic tool, the Lorentz transformation, is easily derived from Einstein’s remarkable assumption that the speed of light must be independent of the observer.

The purpose of these notes is to derive a few of the consequence of special relativity without using anything more than a single linear transformation.

1.1 Historical Note

Although classical physics—the theory founded by Galileo and Newton—was shown to be wrong by Einstein, it was sufficiently accurate for most purposes.

Consider a situation that would have been familiar to Newton. A person walks briskly at 6 k/h and a cantering horse passes by at 16 k/h. According to Newton, the speed of the rider, as seen by the walker, is $16 - 6 = 10$ k/h. According to Einstein, this is incorrect: the relative speed is 10.0000000000000167 k/h. We can forgive Newton for not noticing this discrepancy.

Since light travels fast ($c = 299\,792\,458$ m/s), everyday velocities are usually very slow in comparison. This is why we think that relativistic effects are unimportant in everyday life.

These notes are organized for clarity but are not faithful to history. The following is a brief account of the work that led to Einstein’s insight.

By 1870, there was clear evidence that light had wave-like properties. Maxwell, searching to unify current knowledge of electric and magnetic fields, came up with a set of equations (now called, not surprisingly, “Maxwell’s equations”) that predicted electromagnetic radiation travelling at the speed of light. This strongly suggested that light was a form of electromagnetic radiation, as we now know it is.

Since Maxwell’s equations *predict* the value of c , they can be *applied* only when that value is correct. In other words, it seemed that Maxwell’s equations applied only to a system at rest in the “aether”. In terms of the discussion above, Maxwell’s equations were not invariant under Galilean transformation.

Michelson and Morley attempted to measure the velocity of the earth with respect to the supposed aether, assuming that the speed of light would depend on the direction in which it was measured. They failed to find any such dependence. Lorentz and FitzGerald (1890s) suggested that the discrepancy between the Galilean transformation and Maxwell’s laws could be resolved by assuming that objects shrink as they move through the aether; the shrinkage is just enough to ensure that we cannot detect motion with respect to the aether. The shrinkage is described mathematically by the *Lorentz transformation*.

Lorentz noticed that Maxwell’s equations *were* invariant under his transformation. Thus there appeared to be one set of physical laws for mechanics (Galilean) and another for electromagnetic phenomena (Lorentz). Einstein (1905) suggested, instead, that the Lorentz transformation applied to *all* phenomena, not just electromagnetism. He claims that the Michelson-Morley experiment (failing to detect motion through the aether) did not influence him; instead, he studied electromagnetic theory, *via* Maxwell’s equations, and asked himself such questions as: “What would I see if I travelled at the speed of light?” This reasoning led him to two postulates:

- The laws of physics have the same form in all inertial frames of reference.
- Light propagates through empty space with a velocity that is independent of the observer.

Einstein realized that the Lorentz transformation is consistent with these postulates, as we have already seen. The Special Theory of Relativity says that the laws of physics (mechanics and electromagnetism) are invariant under the Lorentz Transformation.

1.2 Measuring the speed of light

Let's measure the speed of light. We have a source of light and a clock. We set up a mirror at distance L from the light. The measurement consists of turning on the light and starting the clock simultaneously and then waiting until we see the light reflected in the mirror. Suppose that the time that elapses between turning on the light and seeing its reflection is T . Then our estimate of the speed of light, c , is

$$c = \frac{2L}{T}.$$

This is classical physics, as understood by Galileo, Newton, and others. It is helpful to introduce a concept that they would have understood: the event. An *event* is an occurrence that can be determined by its position in space and a time. Working in one dimension, as we will throughout this note, we write (x, t) to denote an event that occurs at position x and time t .

In the experiment above, there are three events:

Turning the light on:	$(0, 0)$
The light reaches the mirror:	$(L, L/c)$
The light returns to the starting point:	$(0, 2L/c)$

Suppose that someone else is observing our experiment from a passing train. The train moves with uniform velocity v in the X direction. In classical physics, the observer's coordinates (x', t') are related to our coordinates (x, t) by the *Galilean transformation*

$$\begin{aligned}x' &= x - vt \\t' &= t.\end{aligned}$$

To keep things simple, we have assumed that our frame of reference and the observer's have the same origin: our event $(0, 0)$ transforms to the observer's event $(0, 0)$.

Transforming the three events above into the observer's reference frame, we obtain:

	(x, t)	\longrightarrow	(x', t')
Turning the light on:	$(0, 0)$	\longrightarrow	$(0, 0)$
The light reaches the mirror:	$(L, L/c)$	\longrightarrow	$(L - vL/c, L/c)$
The light returns to the starting point:	$(0, 2L/c)$	\longrightarrow	$(-2vL/c, 2L/c)$

The observer estimates the velocity of light for the outward part of the path by dividing the observed distance travelled by the observed elapsed time:

$$\frac{L - vL/c}{L/c} = c - v,$$

And for the return journey:

$$\frac{-2vL/c - L + vL/c}{2L/c - L/c} = -c - v.$$

This is precisely what a classical observer would expect: the *apparent* velocity of light is obtained by subtracting v from c when the train is moving in the same direction as the light and by adding v and c when the train is moving in the opposite direction to the light.

Unfortunately, the prediction does not agree with observation. The reasoning is sound, and the experiment would give the expected result if we replaced the light by a gun, the mirror by a sound reflector, and do the experiment using sound rather than light. But, as experiments by Michelson and Morley in the late nineteenth century showed, light does not behave like this. The problem is that the Galilean transformation doesn't work.

2 The Lorentz Transformation

One of Einstein's several great insights into 1905 was to realize that a different transformation was required to explain the phenomena. The point of this note is that, although Einstein's insight was extraordinary and changed the direction of physics, the transformation that we need can be deduced by accepting Einstein's premises and performing some simple reasoning.

The premise that we need is this: *the laws of physics are independent of the observer*. So, for example, the apparent speed of light should not depend on whether you are standing in a station or on a moving train. All attempts to measure it should yield c , not $c \pm v$.

In these notes, we will assume that the various observers may be moving with respect to one another, but they are not accelerating with respect to one another.¹

Still working in one dimension, the transformation we are looking for will have the form

$$\begin{aligned}x' &= X(x, t, v) \\t' &= T(x, t, v)\end{aligned}$$

We will assume that the transformation is *linear* in x and t . There is no real justification for this except that a non-linear transformation would have extremely weird effects on the observable behaviour of the universe and is therefore somewhat unlikely. We will therefore write the transformation from frame S to frame S' as

$$\begin{aligned}x' &= Ax + Bct + x'_0 & (1) \\ct' &= Cx + Dct + ct'_0 & (2)\end{aligned}$$

in which the coefficients A , B , C , and D may depend on v , the relative velocity of the frames, but do not depend on x or t . Moreover:

- We use c to denote the velocity of light. By using ct rather than simply t in the equations, we ensure that each term has the dimensions of a length and that the four constants A , B , C , and D that we have introduced are pure numbers (that is, have no dimensions). The reason for using c as a “standard” velocity is that it appears to be a fundamental constant of the universe.
- The origin of S , $(0, 0)$, transforms to (x'_0, t'_0) . We can simplify things without losing generality by making the origins coincide. Thus we will assume that $x'_0 = t'_0 = 0$ and that the event $(0, 0)$ in S transforms to $(0, 0)$ in S' .

A observer O' in system S' looks at a ray of light passing through the origin and observes that it satisfies the equation

$$x' = ct'.$$

According to (1) and (2), a observer O in system S looking at the same ray will see that it satisfies the equation

$$Ax + Bct = Cx + Dct$$

which we may write as

$$(A - C)x = (D - B)ct.$$

¹Contrary to widespread belief, special relativity can handle acceleration in simple cases. We need general relativity only to calculate accelerations due to the presence of mass.

Since the velocity of light is c for all observers, this implies

$$A - C = D - B. \quad (3)$$

The observer O' now looks at a ray going in the opposite direction and sees that it satisfies

$$x' = -ct'.$$

To observer O , this ray satisfies

$$Ax + Bct = -Cx - Dct$$

and O infers

$$A + C = B + D. \quad (4)$$

The sum of (3) and (4) gives $A = D$ and their difference gives $B = C$. Thus we can simplify the original transformation equations (1) and (2) to

$$x' = Ax + Bct \quad (5)$$

$$ct' = Bx + Act \quad (6)$$

At the space origin of S' , we have $x' = 0$ and, from (5),

$$Ax + Bct = 0$$

or

$$\frac{x}{t} = -\frac{Bc}{A}$$

This suggests that the relative velocity of the two systems is

$$v = -\frac{Bc}{A}$$

and further evidence of this is obtained by noting that, at the origin of S , where $x = 0$, we have

$$x' = Bct$$

$$ct' = Act$$

and therefore

$$\begin{aligned} \frac{x'}{t'} &= \frac{Bc}{A} \\ &= -v \end{aligned}$$

Thus O and O' can agree that their systems are moving past each other with velocity v . We can now write the transformation as

$$x' = Ax - Avt \quad (7)$$

$$ct' = -Avx/c + Act \quad (8)$$

We suppose that there is a measuring rod of length L , fixed with respect to S , and with one end at the origin. We define events for the ends of the rod as:

$$E_0 = (0, 0) \quad (9)$$

$$E_L = (L, ct) \quad (10)$$

where t is to be determined later. Note that we are using ct rather than t as the time coordinate, so that an event is a pair of lengths.

For the observer O' in S' , events (9) and (10) are transformed by (7) and (8) to

$$E'_0 = (0, 0) \quad (11)$$

$$E'_L = (AL - Avt, -AvL/c + Act) \quad (12)$$

To find out what O' sees at time $t' = 0$, we solve

$$-AvL/c + Act = 0$$

for t , obtaining $t = vL/c^2$. Substituting this value into (11) and (12) gives

$$E'_0 = (0, 0)$$

$$E'_L = (AL(1 - v^2/c^2), 0)$$

Thus observer O' sees the length of the rod as $AL(1 - v^2/c^2)$.

We now reverse the roles. There is a rod of length L with one end at the origin of S' . The events corresponding to the ends of this rod are

$$E'_0 = (0, 0) \quad (13)$$

$$E'_L = (L, ct') \quad (14)$$

Transforming (13) and (14) back into S gives $E_0 = (0, 0)$ and, for E_L ,

$$L = Ax - Avt \quad (15)$$

$$ct' = -Avx/c + Act \quad (16)$$

At $t = 0$, (15) and (16) give $E_L = (L/A, 0)$, showing that O sees the rod as having length L/A .

According to the principle of relativity, there should be no difference between the observations of O and O' . It follows that the apparent lengths of the rods must be the same. Thus

$$L/A = LA(1 - v^2/c^2)$$

and so

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Substituting this value into (7) and (8) gives the *Lorentz transformation*:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$ct' = \frac{ct - vx/c}{\sqrt{1 - v^2/c^2}}$$

2.1 Notation

Following convention, we define

$$\beta \stackrel{\text{def}}{=} \frac{v}{c}$$

$$\gamma \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 - v^2/c^2}}.$$

It is useful to remember that $\beta < 1$ and $\gamma > 1$, both of which follow immediately from $v < c$. Using this convention, we can write the Lorentz transformation in equational form as

$$\begin{aligned} x' &= \gamma x - \gamma\beta ct \\ ct' &= -\gamma\beta x + \gamma ct \end{aligned} \tag{17}$$

and in matrix form as

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \times \begin{bmatrix} x \\ ct \end{bmatrix}$$

The determinant of the transformation matrix is

$$\begin{vmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{vmatrix} = \gamma^2 - \gamma^2\beta^2 = 1$$

showing that the Lorentz transformation is analogous to a rotation, but in hyperbolic space-time rather than in familiar Euclidean space (more in Section 5).

The inverse transformation is easily found by changing the sign of v :

$$\begin{aligned} \gamma x' + \gamma\beta ct' &= (\gamma^2 x - \gamma^2\beta ct) + (-\gamma^2\beta^2 x + \gamma^2\beta t) = x \\ \gamma\beta x' + \gamma ct' &= (\gamma^2\beta x - \gamma^2\beta^2 ct) + (-\gamma^2\beta x + \gamma^2 ct) = ct \end{aligned}$$

Furthermore, we find that

$$\begin{aligned} c^2 t'^2 - x'^2 &= (-\gamma\beta x + \gamma ct)^2 - (\gamma x - \gamma\beta t)^2 \\ &= \gamma^2 (\beta^2 x^2 - 2vxt + c^2 t^2 - x^2 + 2vxt - v^2 t^2) \\ &= \gamma^2 [c^2 t^2 (1 - \beta^2) - x^2 (1 - \beta^2)] \\ &= c^2 t^2 - x^2 \end{aligned} \tag{18}$$

showing that $c^2 t^2 - x^2$ is an *invariant* of the Lorentz transformation.

In relativistic physics, we should not talk about “stationary” and “moving”. For brevity, however, we adopt the following conventions in these notes:

a *stationary observer* is an observer who is fixed with respect to the reference frame of the objects we are talking about;

a *moving observer* is an observer who is moving along the X axis of the reference frame of the objects we are talking about.

When we write coordinates, we put the space dimension first, as in (x, ct) .

3 Applications

3.1 Separations and Light Cones

The invariant $c^2t^2 - x^2$ of (18) may be positive or negative. In general, given two events $E_1 = (x_1, ct_1)$ and $E_2 = (x_2, ct_2)$, we define their *separation* ds by

$$ds^2 = (ct_1 - ct_2)^2 - (x_1 - x_2)^2$$

and classify the separation by the sign of ds^2 :

Spacelike separation: $ds^2 < 0$ and ds is imaginary. No observer can be present at both events and neither event can affect the other, since to do so would require the cause to travel faster than light.

Lightlike separation: $ds^2 = 0$ and ds is—obviously—zero. Only a photon can be present at both events.

Timelike separation: $ds^2 > 0$ and ds is real. An observer travelling at less than the speed of light could be present at both events.

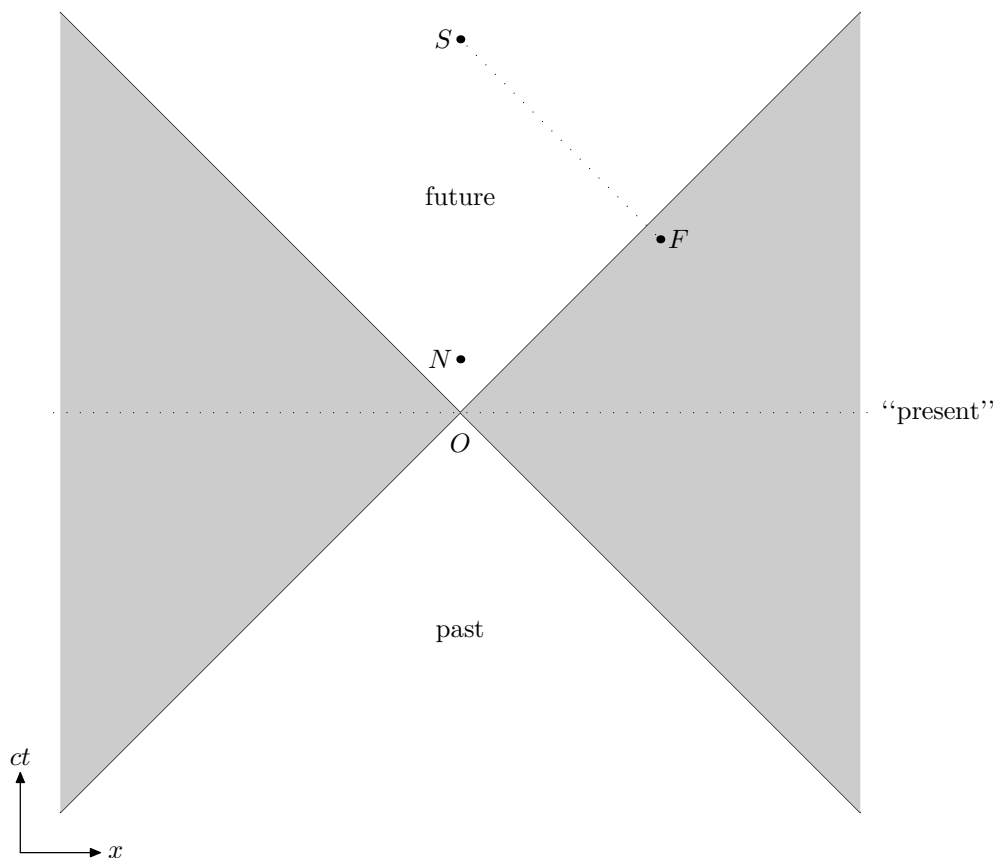


Figure 1: Past, future, and unknown

Figure 1 illustrates separation with respect to an observer at O . The lines at 45° correspond to light rays and contain all points with lightlike separation from O , the shaded regions contain

all points with spacelike separation, and the clear regions contain all points with timelike separation. The clear regions are past events that may have affected O in some way and future events that O may affect in some way.

The dotted line labelled “present” suggests the rather strange concept of “the present” in special relativity. It consists of all the events that occur “now” in the observer’s reference frame. However, *none* of these events, except O itself, is accessible to the observer, because their separation from the observer is spacelike.

Like most figures in these notes, Figure 1 is a *spacetime diagram* in which time ct increases upwards (i.e., from the bottom of the page to the top) and the single space dimension, x , increases from left to right.

If we added a space dimension to the diagrams, which would then have x , y , and ct dimensions, the clear regions of Figure 1 would form a cone with its apex at the observer, O . For this reason, the four-dimensional regions of the actual universe that are accessible to an observer are called the observer’s *light cones*.

Two events can be observed as simultaneous to some observer if and only if their separation is spacelike. Suppose that the two events in the stationary reference frame are $E_1 = (x_1, ct_1)$ and $E_2 = (x_2, ct_2)$. For a moving observer:

$$E'_1 : \begin{aligned} x'_1 &= \gamma x_1 - \gamma \beta ct_1 \\ ct'_1 &= -\gamma \beta x_1 + \gamma ct_1 \end{aligned} \qquad E'_2 : \begin{aligned} x'_2 &= \gamma x_2 - \gamma \beta ct_2 \\ ct'_2 &= -\gamma \beta x_2 + \gamma ct_2 \end{aligned}$$

The moving observer will see the events simultaneously if $ct'_1 = ct'_2$, or

$$-\gamma \beta x_1 + \gamma ct_1 = -\gamma \beta x_2 + \gamma ct_2$$

which simplifies to

$$\beta(x_1 - x_2) = c(t_1 - t_2). \tag{19}$$

Thus

$$\begin{aligned} ds^2 &= (ct_1 - ct_2)^2 - (x_1 - x_2)^2 \\ &= (x_1 - x_2)^2 (\beta^2 - 1) \end{aligned}$$

Since $\beta < 1$, ds^2 must be negative, showing that E_1 and E_2 must be spacelike separated.

It is not the case that we can never become aware of events outside our light cones. In Figure 1, suppose that an astronaut sets off a firework on the moon at noon earth-time (event F). The observer, one second before noon on earth (event N) cannot influence the explosion in any way, because the moon is about $1\frac{1}{4}$ light-seconds away from the earth. Nevertheless, the observer will eventually discover that the firework has exploded, when light (dotted line) from it reaches the earth (event S).

According to (19), an observer, passing by in a spaceship at velocity v , would see N and F as simultaneous events if

$$\frac{v}{c} = c \frac{t}{d}$$

where d is the distance of the moon from the earth and t is the time difference (in the earth’s frame) between the events. We have

$$\begin{aligned} \frac{v}{c} &\approx 3 \times 10^8 \times \frac{4 \times 10^8}{1} \\ &= 0.75. \end{aligned}$$

3.2 Time Dilation and Space Contraction

Consider two events in the stationary frame, occurring at the same place but at different times. For simplicity, assume the coordinates of the events are $E_1 = (0, 0)$ and $E_2 = (0, cT)$.

In the moving observer's frame, the corresponding coordinates are $E'_1 = (0, 0)$ and $E'_2 = (-\beta cT, \gamma cT)$. The second event is displaced by $\beta cT = vT$, as a good Newtonian would expect, but the time difference between the two events is γcT , which is longer ($\gamma > 1$) than cT .

The transformed value of x in the Lorentz transformation is $x' = \gamma x - \gamma\beta ct$, which suggests that space is dilated ($\gamma x > x$) as well. But this is misleading. To see why, we consider two points in the stationary frame, P_0 at the origin and P_d at distance d from the origin.

The world line of P_0 is $x = 0$ for all t . The moving observer sees

$$\begin{aligned}x'_0 &= -\gamma\beta ct \\ ct' &= \gamma ct.\end{aligned}$$

Eliminating t from these equations gives the world line of P'_0 as

$$x'_0 = -\beta ct'.$$

The world line of P_d is $x = d$ for all t . The moving observer sees

$$\begin{aligned}x'_d &= \gamma d - \gamma\beta ct \\ ct' &= -\gamma\beta d + \gamma ct.\end{aligned}$$

Eliminating t gives the world line of P'_d as

$$x'_d = -\beta ct' + d/\gamma.$$

At time t' , we have

$$\begin{aligned}x'_d - x'_0 &= -\beta ct' + d/\gamma + \beta ct' \\ &= d/\gamma,\end{aligned}$$

showing that the distance is independent of time, as we would hope, but that the distance between the points, as seen by the moving observer, is $d/\gamma < d$.

These are well-known facts of relativity: to a moving observer, time passes more slowly (time dilation), and distances in the direction of motion are smaller (space contraction).

If an object moved past you very rapidly, would you actually see it foreshortened in the direction of motion, as the Lorentz transformation apparently predicts? This is quite a different question! What the Lorentz transformation says is that, if you record the coordinates of the front and back of the moving object *at a particular time*, the distance between them will be less than that measured by an observer travelling with the object. What you would *see*, however, depends on the time that light takes to travel from the object to your eyes. usually, of course, this time is quite insignificant and so we ignore it but, when objects are moving near light speed, it becomes important. We will not investigate this issue further here² except to note that it is possible that the object appears rotated—an effect called *Penrose-Terrell rotation*.³

²But watch this space!

³<http://math.ucr.edu/home/baez/physics/Relativity/SR/penrose.html>.

As an example of time dilation in everyday life, consider a muon entering the earth's atmosphere with a speed of $0.99c$. We observe that the muon travels 4632 m before it decays. Since the average lifetime of a muon is $2.2 \mu\text{s}$, it appears that its velocity is $4632/(2.2 \times 10^{-6}) = 21 \times 10^8 \text{ m/s}$ —seven times the speed of light!

The interpretation according to special relativity is as follows. The muon has an average lifetime of $2.2 \mu\text{s}$, so we can model its life by two events: birth at $(0, 0)$ and death at $(0, 2.2 \times 10^{-6})$ in a system with its origin at the centre of the muon. Applying the Lorentz transformation (see Section 2) with $v = 0.99c$ gives the transformed birth and death events as $(0, 0)$ and $(4632, 15.6 \times 10^{-6})$ respectively. The $2.2 \mu\text{s}$ lifetime of the muon expands to $15.6 \mu\text{s}$ as it travels 4632 m.

3.3 Reflections

Figures 2, 3, and 4 show three views of the same physical events. Three objects are indicated by their world lines: PP' is the world line of a light source and detector; AA' and BB' are the worldlines of mirrors.

The events are as follows: a light flashes briefly at P , sending rays to the mirrors. The light reaches mirror A at R and mirror B at S . The mirrors reflect the light back to the source P .

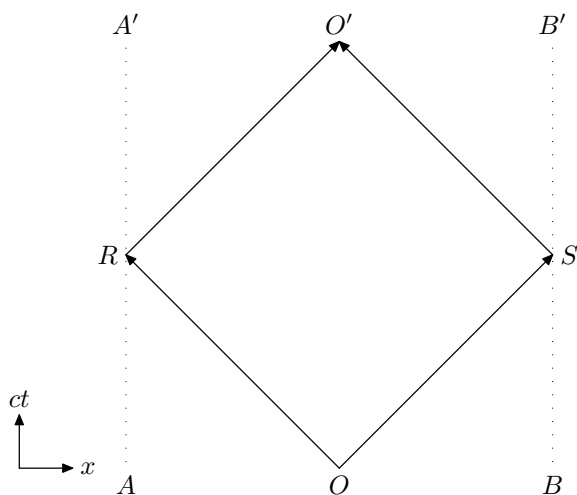


Figure 2: Reflections with $v = 0$

The figures show the events as seen by observers travelling at different velocities with respect to P . They show that:

- The speed of light is always c relative to the observer. In each figure, the arrows representing light rays have slope ± 1 .
- The left-moving ray and the right-moving ray leave P simultaneously and their reflections return to P simultaneously (otherwise things would be really peculiar!).
- However, the time between the reflection events at R and S *does* depend on the motion of the observer. For the observer at rest, they are simultaneous. For the moving observers, S occurs before R .

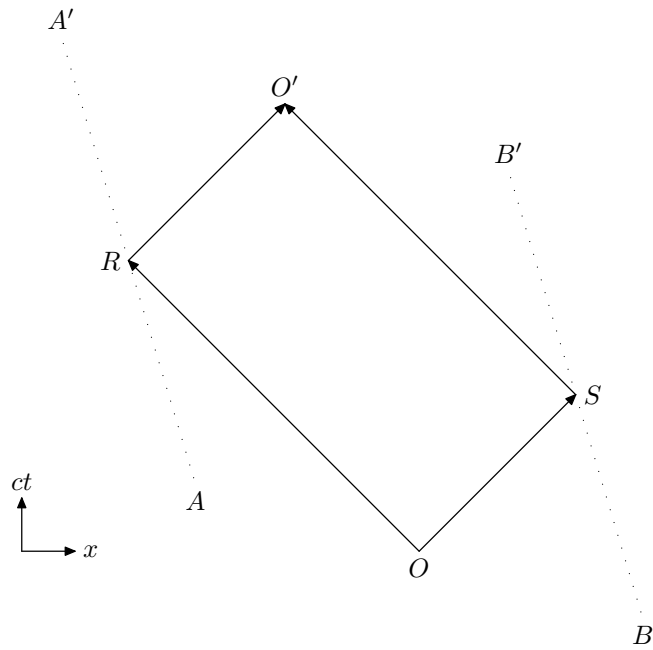


Figure 3: Reflections with $v = 0.3c$

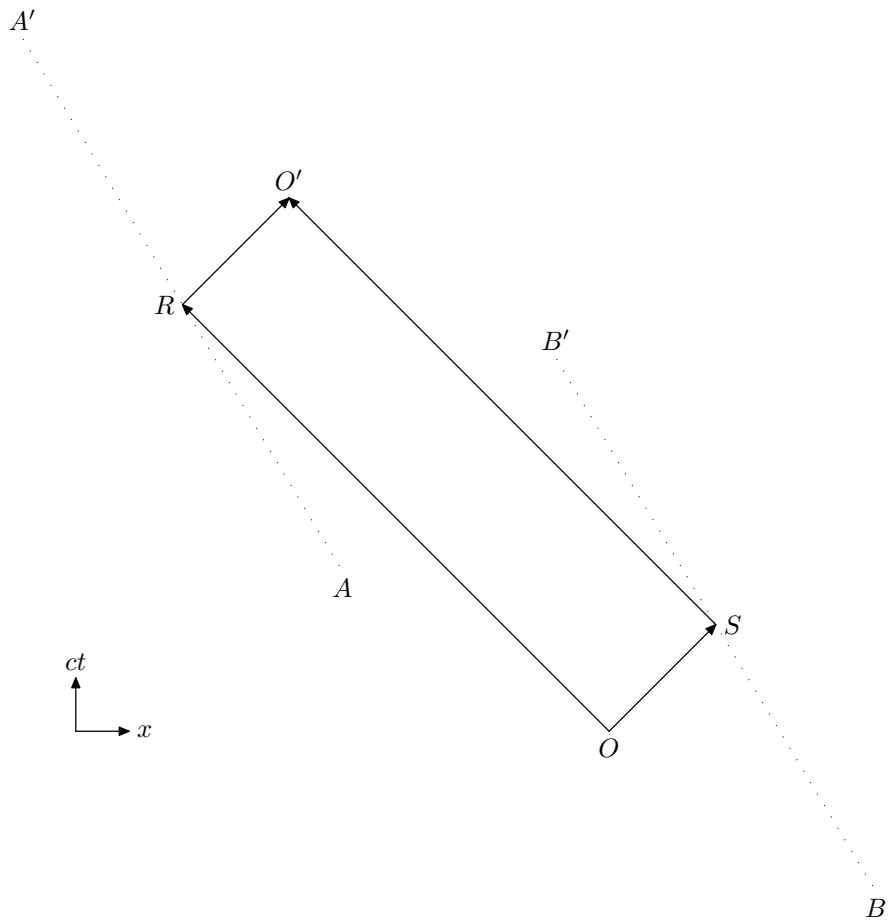


Figure 4: Reflections with $v = 0.6c$

- Motion foreshortens distance: the horizontal distance (i.e., distance at a given time) between AA' and BB' is greatest for the stationary observer, and decreases as the relative velocity of the observer increases.
- Motion dilates time: the time at P' is least for the stationary observer and increases with the relative velocity of the observer.

The following table shows the coordinates of the events as seen by an observer moving with velocity v relative to P . As usual, we let $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. For the stationary observer, $\beta = 0$ and $\gamma = 1$. L is the distance between the lamp and each mirror.

Event	Stationary observer		Moving observer	
	x	ct	x'	ct'
P	0	0	0	0
R	$-L$	L	$-\gamma(1 + \beta)L$	$\gamma(1 + \beta)L$
S	L	L	$\gamma(1 - \beta)L$	$\gamma(1 - \beta)L$
P'	0	$2L$	$-2\gamma\beta L$	$2\gamma L$

The next table shows the equations of various lines, as seen by the stationary observer and the moving observer.

Line	Stationary observer	Moving observer
PP'	$x = 0$	$x' = -\beta ct'$
AA'	$x = -L$	$x' = -\beta ct' - L/\gamma$
BB'	$x = L$	$x' = -\beta ct' + L/\gamma$
PR	$x = -ct$	$x' = -ct'$
PS	$x = ct$	$x' = ct'$
RP'	$x = ct - 2L$	$x' = ct' - 2\gamma(1 - \beta)L$
SP'	$x = -ct + 2L$	$x' = -ct' + 2\gamma(1 - \beta)L$

The next table shows the actual values used to draw Figures 2, 3, and 4 with $L = 100$.

Event	$v = 0$		$v = 0.5c$		$v = 0.8c$	
	x	ct	x	ct	x	ct
$A B$	± 100	0	± 86	0	± 60	0
R	-100	100	-173	173	-300	300
S	100	100	58	58	33	33
P'	0	200	-115	230	-267	333

3.4 Clocks

Continuing the theme of reflection, we consider a clock that is conceptually simple but hard to build. The clock consists of two parallel mirrors, separated by a distance d , and a photon that bounces between them. The period of the clock is $2d/c$, the time taken for the photon to travel from one mirror to the other and back again. The frequency of the clock is the reciprocal of the period: $f = c/2d$. If we place the mirrors 5 cm apart, the frequency of the clock will be approximately $3 \text{ GHz} = 3 \times 10^9$ cycles per second, in the same range as a modern PC clock.

We refer to an event in which the photon bounces off the left mirror as a “tick” and to an event in which it bounces off the right mirror as a “tock”.

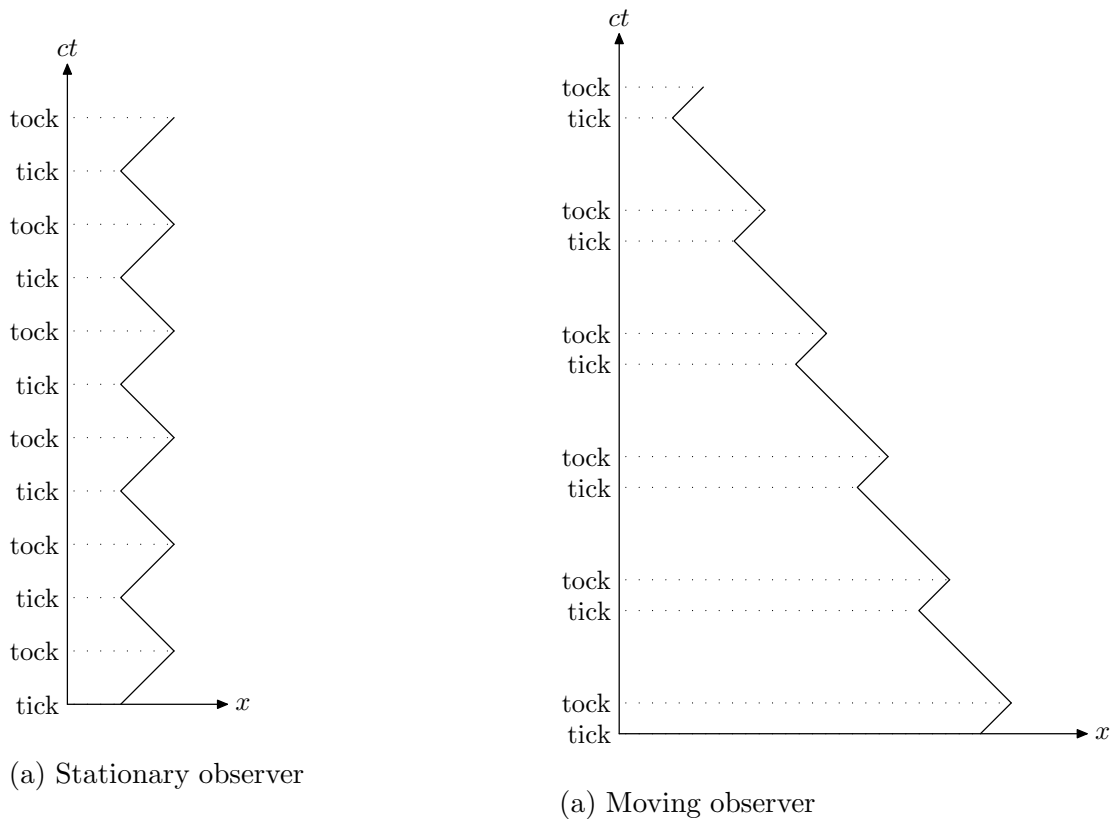


Figure 5: The photon clock as seen by two observers

The spacetime diagram at the left of Figure 5 shows the clock as seen by a stationary observer. We note that the clock emits a steady sequence of ticks and tocks.

A moving observer sees the clock in a rather different way, as shown by the spacetime diagram at the right of Figure 5. The clock is smaller, its period is longer, and the tick-tock interval is smaller than the tock-tick interval. The differences are summarized in the following table.

	Stationary observer	Moving observer
Distance between mirrors	d	d/γ
Clock period	$2d/c$	$2\gamma d/c$
Time from tick to tock	d/c	$\gamma d(1 - \beta)/c$
Time from tock to tick	d/c	$\gamma d(1 + \beta)/c$
n th tick event	$(0, 2nd)$	$(-2n\gamma\beta d, 2n\gamma d)$
n th tock event	$(d, (2n + 1)d)$	$(\gamma d - \gamma\beta(2n + 1)d, \gamma(2n + 1)d - \gamma\beta d)$

3.5 The Bull and the Barn

This story is usually called “the pole and the barn”, but replacing the pole by a bull adds to the drama.

A farmer has a large bull that has escaped. There is a barn directly in the path of the bull, and the farmer reasons that if he closes the doors at both ends of the barn while the bull is passing through, the bull will be trapped. Unfortunately, the bull is longer than the barn.

A farmhand points out that the bull is running so fast that the Lorentz contraction will enable it to fit inside the barn. The farmer is doubtful but agrees to a preliminary test with flashlights. They put a switch in the middle of the barn, wired to lights at each end, so that the lights can be flashed simultaneously.⁴

This experiment gives the results shown in Figure 6, in which the world lines are: FF' , the front of the barn; BB' , the back of the barn; HH' , the head of the bull; and TT' , the tail of the bull. The flashes are shown as asterisks. The bull is running at $0.8c$. The farmer notes that it is indeed the case that, when the lights flash, both the head and tail of the bull are inside the barn.

The bull sees things slightly differently. The solid lines in Figure 6 join the bulls' head to the bull's tail in the bull's frame of reference. Since there is no solid line within the barn, the bull disagrees with the farmer: at no time is it entirely in the barn.

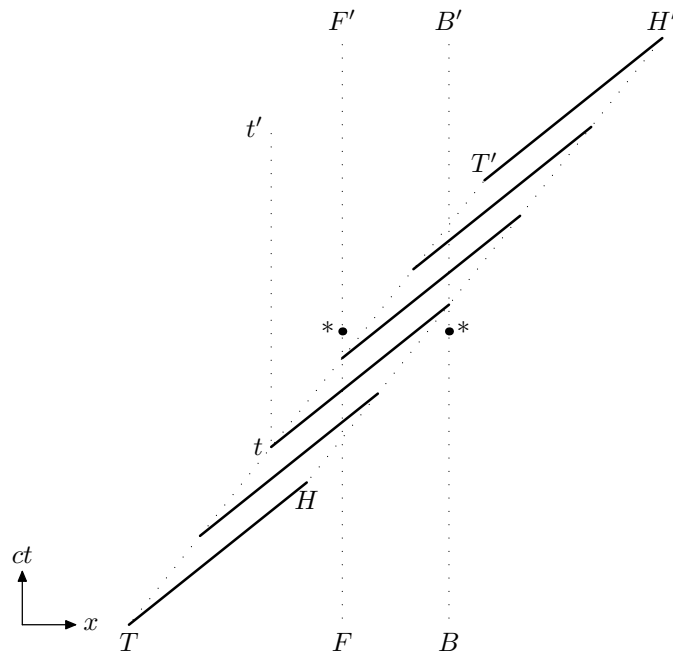


Figure 6: Standing in the barn

Figure 7 shows the scene in the bull's frame of reference. The bull sees the flash at the front of the barn before his head reaches the back of the barn, and the flash at the back of the barn after his tail has passed the front of the barn. However, by that time, his head is well outside the other end of the barn.

What would happen if the farmer decided that the flashing experiment was successful and replaced the lights with automatic doors? The farmer would close the doors simultaneously. The bull would enter the barn and see the barn's back door close while his tail was still outside. His head would hit the door . . . forces and accelerations enter the picture and things become rather complicated. To avoid the complexity, assume that, when the bull's head strikes the barn door, the entire bull comes to rest instantly. The world line of the bull's tail turns abruptly, becoming parallel to the barn's world lines: it is shown as tt' in both diagrams. With this scenario, the bull's tail is outside the barn when the front door closes.

⁴In the barn's reference frame, of course.

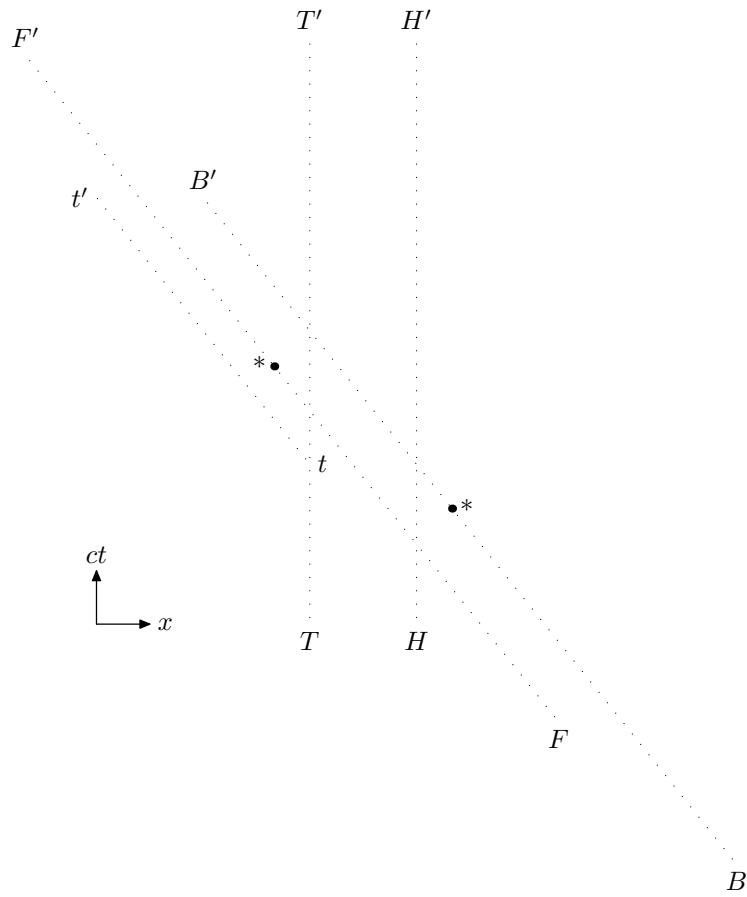


Figure 7: Running with the bull

What does the farmer make of this? The bull’s world lines turn abruptly in his diagram, too. To the farmer, it appears that the bull’s tail stops moving while it is still outside the barn and *before he has closed the barn doors*. This is just an example of super-luminal communication (from the bull’s head to its tail) giving rise to effects that precede their causes.

In practice, impact of the bull’s head against the door could not be communicated instantaneously to the bull’s tail. Some compression of the bull would undoubtedly take place, or perhaps the door would give way, but further discussion of the resulting effects⁵ is beyond the scope of these notes.

3.6 Space Travel

A journey by spaceship to a distant star (or anywhere else) can be described by two events in a frame fixed with respect to the origin and destination. The events are the start of the journey, $E_s = (0, 0)$, and the end (or “finish”) of the journey, $E_f = (D, T)$, where D is the distance of the star and T is the time it takes to get there. The corresponding events in the spaceship, assumed to travel with uniform velocity v , are

$$\begin{aligned} E'_s &= (0, 0) \\ E'_f &= (D', T') \end{aligned}$$

where

$$\begin{aligned} D' &= \gamma D - \gamma \beta c T \\ c T' &= -\gamma \beta D + \gamma c T \end{aligned}$$

Since $D' = 0$, we infer

$$T = \frac{\gamma D}{\gamma \beta c} = \frac{D}{v}$$

and therefore

$$\begin{aligned} c T' &= -\gamma \beta D + \gamma c D / v \\ &= \frac{c D}{v} \sqrt{1 - \frac{v^2}{c^2}}. \end{aligned}$$

Consequently,

$$T' = \frac{D}{v} \sqrt{1 - \frac{v^2}{c^2}}, \tag{20}$$

and, by making v large, we can make T' arbitrarily small. In fact, solving (20) for v gives a formula that enables us to find the velocity required to travel a given distance in a given time:

$$v = \frac{c}{\sqrt{1 + c^2 T^2 / D^2}}.$$

Barnard’s star, the closest to us, is 4 light-years away. To get there in 1 year, as measured by travellers on the space ship, the spaceship would have to travel at $0.97c$. To get there in a day, it would have to travel at $0.999999765c$.

⁵Since the bull is moving 250,000 times faster than a rifle bullet, these effects could be quite dramatic.

3.7 Aging Twins

In the “twin paradox”, one twin stays home while the other travels and returns. When the twins meet again, the traveller is younger than the stay-at-home.

We model this adventure with three frames:

- $F^{(0)}$ $\stackrel{\text{def}}{=}$ the “stationary” frame of the twin who stays at home;
- $F^{(v)}$ $\stackrel{\text{def}}{=}$ the outgoing frame, moving with velocity v with respect to $F^{(0)}$; *and*
- $F^{(-v)}$ $\stackrel{\text{def}}{=}$ the returning frame, moving with velocity $-v$ with respect to $F^{(0)}$.

We will use column vectors for events. The frames have a common origin, the event $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Event names have the form E_n^f , in which f indicates the frame, using the same notation as for frames, and n is a sequence number. The frames have a common origin, the event

$$E_0^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

At zero time, in all frames, one twin steps onto frame $F^{(v)}$. The corresponding event is

$$E_0^{(v)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The next event occurs when the traveller has travelled for time T in $F^{(v)}$: it is

$$E_1^{(v)} = \begin{bmatrix} 0 \\ T \end{bmatrix}.$$

At this time, the twin steps onto frame $F^{(-v)}$. To find the corresponding event in frame $F^{(-v)}$, we apply the transformation obtained by composing the transformations for v and $-v$:

$$\begin{aligned} E_1^{(-v)} &= \begin{bmatrix} \gamma^2(1 + \beta^2) & 2\beta\gamma^2 \\ 2\beta\gamma^2 & \gamma^2(1 + \beta^2) \end{bmatrix} \begin{bmatrix} 0 \\ T \end{bmatrix} \\ &= \begin{bmatrix} 2\beta\gamma^2 T \\ \gamma^2(1 + \beta^2)T \end{bmatrix}. \end{aligned}$$

The twin stays in this frame for a further time T , ending at

$$E_2^{(-v)} = \begin{bmatrix} 2\beta\gamma^2 T \\ \gamma^2(1 + \beta^2)T + T \end{bmatrix}.$$

The stay-at-home sees this event in frame $F^{(0)}$:

$$\begin{aligned} E_2^{(0)} &= \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 2\beta\gamma^2 T \\ \gamma^2(1 + \beta^2)T + T \end{bmatrix} \\ &= \begin{bmatrix} 2\beta\gamma^3 T - \beta\gamma T(1 + \gamma^2(1 + \beta^2)) \\ -2\beta^2\gamma^3 T + \gamma T(1 + \gamma^2(1 + \beta^2)) \end{bmatrix} \\ &= \begin{bmatrix} T\beta\gamma(2\gamma^2 - 1 - \gamma^2 - \gamma^2\beta^2) \\ \gamma T(-2\beta^2\gamma^2 + 1 + \gamma^2 + \gamma^2\beta^2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2\gamma T \end{bmatrix}. \end{aligned}$$

The space coordinate, 0, shows that the traveller has returned home, as expected. The time coordinate, $2\gamma T$, shows that the time in the home frame is slightly later than the traveller's time. While the traveller has aged $2T$, the stay-at-home twin has aged $2T/\sqrt{1-v^2/c^2}$.

There is no paradox. In Euclidean geometry, we have the *triangle inequality*: the sum of the lengths of two sides of a triangle is greater than the length of the third side. In Minkowski space-time, the sum of the lengths of two sides of a triangle is *less* than the length of the third side, provided we interpret "length" as "time measured by a clock travelling with the frame".

The abrupt change of velocity from $E_1^{(v)}$ to $E_1^{(-v)}$ is physically impossible. A smooth transition (decelerating to zero velocity with respect to $F^{(0)}$, the accelerating in the opposite direction) does not change the conclusion—it just makes the calculation harder.

The terrestrial effects of the twin paradox are quite small. A return flight across the Atlantic will extend your life, relative to non-travellers, by about 18 nanoseconds.

4 Adding Velocities

Suppose that a frame S_u is moving with velocity u relative to the observer's frame S_o and frame S_v is moving with velocity v with respect to S_u . Let

$$\begin{aligned}\beta_u &\stackrel{\text{def}}{=} u/c & \gamma_u &\stackrel{\text{def}}{=} \sqrt{\frac{1}{1-u^2/c^2}} \\ \beta_v &\stackrel{\text{def}}{=} v/c & \gamma_v &\stackrel{\text{def}}{=} \sqrt{\frac{1}{1-v^2/c^2}}\end{aligned}$$

Then the transformation from S_o to S_v is

$$\begin{bmatrix} \gamma_u & \gamma_u \beta_u \\ \gamma_u \beta_u & \gamma_u \end{bmatrix} \times \begin{bmatrix} \gamma_v & \gamma_v \beta_v \\ \gamma_v \beta_v & \gamma_v \end{bmatrix} = \begin{bmatrix} \gamma_u \gamma_v + \gamma_u \beta_u \gamma_v \beta_v & \gamma_u \gamma_v \beta_u + \gamma_u \gamma_v \beta_v \\ \gamma_u \gamma_v \beta_u + \gamma_u \gamma_v \beta_v & \gamma_u \gamma_v + \gamma_u \beta_u \gamma_v \beta_v \end{bmatrix}$$

By analogy with the basic transformation, we see that the resultant velocity, w , is $\beta_w c$ where

$$\begin{aligned}\beta_w &\stackrel{\text{def}}{=} w/c \\ \gamma_w &\stackrel{\text{def}}{=} \sqrt{\frac{1}{1-w^2/c^2}} \\ \gamma_w \beta_w &= \gamma_u \gamma_v \beta_u + \gamma_u \gamma_v \beta_v \\ &= \gamma_u \gamma_v (\beta_u + \beta_v) \\ &= \gamma_u \gamma_v (u + v)/c.\end{aligned}$$

and so

$$\begin{aligned}w \gamma_w &= c \gamma_u \gamma_v (u + v)/c \\ &= \gamma_u \gamma_v (u + v).\end{aligned}$$

Squaring gives

$$\begin{aligned}w^2 &= \gamma_u^2 \gamma_v^2 (u + v)^2 (1 - w^2/c^2) \\ &= \gamma_u^2 \gamma_v^2 (u + v)^2 - \frac{w^2}{c^2} \gamma_u^2 \gamma_v^2 (u + v)^2\end{aligned}$$

which we can solve for w^2 , giving

$$\begin{aligned} w^2 &= \frac{c^2 \gamma_u^2 \gamma_v^2 (u+v)^2}{c^2 + c^2 \gamma_u^2 \gamma_v^2 (u+v)^2} \\ &= \frac{(u+v)^2}{D} \end{aligned}$$

where

$$\begin{aligned} D &= \frac{1}{\gamma_u^2 \gamma_v^2} + \frac{(u+v)^2}{c^2} \\ &= \left(1 - \frac{u^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right) + \frac{(u+v)^2}{c^2} \\ &= \left(1 + \frac{uv}{c^2}\right)^2 \end{aligned}$$

Taking square roots gives the final result:

$$w = \frac{u+v}{1+uv/c^2} \quad (21)$$

and repeating the exercise with a change of sign gives the subtraction formulas, for frames moving with velocities u and $-v$:

$$\begin{aligned} w^+ &= (u+v)/(1+uv/c^2) \\ w^- &= (u-v)/(1-uv/c^2). \end{aligned}$$

As we should expect from the derivation of the Lorentz transformation, the result of adding any velocity to c is just c . If $v = c$, then

$$\begin{aligned} w^+ &= (u+c)/(1+u/c) = c \\ w^- &= (u-c)/(1-u/c) = -c. \end{aligned}$$

It is interesting to note that, as $u \rightarrow c$ and $v \rightarrow c$, w^- does not approach zero but, instead, approaches the indeterminate value $0/0$. The relative velocity of two photons in a parallel beam of light, for example, is not zero but is undefined.

5 Hyperbolic Space

Since $\gamma^2 - \gamma^2 \beta^2 = 1$ and $\cosh^2 \theta - \sinh^2 \theta = 1$, we can choose θ such that

$$\begin{aligned} \cosh \theta &= \gamma \\ \sinh \theta &= -\gamma\beta. \end{aligned}$$

With this substitution, the Lorentz transformation matrix becomes

$$\begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}$$

and its inverse becomes

$$\begin{bmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix}.$$

Multiplying these gives

$$\begin{aligned}
& \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix} \times \begin{bmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix} \\
&= \begin{bmatrix} \cosh^2 \theta - \sinh^2 \theta & \sinh \theta \cosh \theta - \sinh \theta \cosh \theta \\ \sinh \theta \cosh \theta - \sinh \theta \cosh \theta & \cosh^2 \theta - \sinh^2 \theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

as expected, and the addition formulas for cosh and sinh give us:

$$\begin{aligned}
& \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix} \times \begin{bmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{bmatrix} \\
&= \begin{bmatrix} \cosh \theta \cosh \phi + \sinh \theta \sinh \phi & \cosh \theta \sinh \phi + \sinh \theta \cosh \phi \\ \cosh \theta \sinh \phi + \sinh \theta \cosh \phi & \cosh \theta \cosh \phi + \sinh \theta \sinh \phi \end{bmatrix} \\
&= \begin{bmatrix} \cosh(\theta + \phi) & \sinh(\theta + \phi) \\ \sinh(\theta + \phi) & \cosh(\theta + \phi) \end{bmatrix}
\end{aligned}$$

Starting from the original transformation, (17),

$$\begin{aligned}
x' &= \gamma x - \gamma \beta ct \\
ct' &= -\gamma \beta x + \gamma ct
\end{aligned}$$

and noting that $\gamma(1 - \beta) = \cosh \theta + \sinh \theta = e^\theta$ and $\gamma(1 + \beta) = \cosh \theta - \sinh \theta = e^{-\theta}$, we can express the Lorentz transformation in a pleasantly symmetric form:

$$\begin{aligned}
x' + ct' &= \gamma(1 - \beta)x + \gamma(1 - \beta)ct = e^\theta(x + ct) \\
x' - ct' &= \gamma(1 + \beta)x - \gamma(1 + \beta)ct = e^{-\theta}(x - ct).
\end{aligned}$$

Introducing the hyperbolic tangent gives

$$\tanh \theta = \frac{\sinh \theta}{\cosh \theta} = \frac{\gamma \beta}{\gamma} = \frac{v}{c}.$$

Define the *rapidity* $r = c\theta$ corresponding to a velocity v by

$$r \stackrel{\text{def}}{=} c \tanh^{-1}(v/c).$$

As v varies in $(-c, c)$, r varies in $(-\infty, \infty)$. Comparing the addition formula for tanh

$$\tanh(\theta + \phi) = \frac{\tanh \theta + \tanh \phi}{1 + \tanh \theta \tanh \phi}$$

with the addition formula for velocities, (21), we see that normal addition of rapidities corresponds to relativistic addition of velocities. In other words, to find the relativistic sum of velocities v_1 and v_2 , first convert them to “rapidity space” using

$$\begin{aligned}
r_1 &= c \tanh^{-1}(v_1/c) \\
r_2 &= c \tanh^{-1}(v_2/c),
\end{aligned}$$

add the rapidities

$$r_{1+2} = r_1 + r_2$$

and convert the result back to a velocity:

$$v_{1+2} = c \tanh(r_{1+2}/c).$$

Figure 8 illustrates the concept of rapidity space. The arithmetic sequence of rapidities,

$$\begin{aligned} r_0 &= 0 \\ r_1 &= r \\ r_2 &= r + r \\ r_3 &= r + r + r \\ &\dots \end{aligned}$$

increases without bound but the corresponding velocities v_1, v_2, v_3, \dots , mapped by \tanh , are bounded by c .

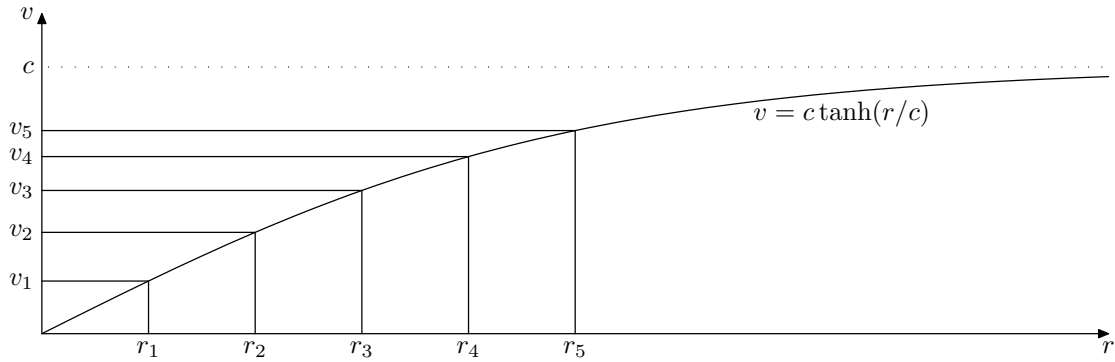


Figure 8: Mapping between rapidity and velocity

In two-dimensional Euclidean space with coordinates X and Y , the transformation

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \times \begin{bmatrix} x \\ t \end{bmatrix}.$$

defines a rotation through angle θ . The origin is a fixed point of this transformation and rotations can be combined by adding arguments:

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \times \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} \cos(\theta + \phi) & \sin(\theta + \phi) \\ -\sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix}$$

Euclidean distance, $\sqrt{x^2 + y^2}$, is an invariant of the transformation.

By analogy, we can view the Lorentz transformation as a rotation in hyperbolic space. In both Euclidean and Lorentzian spaces:

- The origin of the transformation is a fixed point.
- There is no scaling: the determinant of the transform is 1.
- The addition formula combines rotations.
- There is an invariant of the transformation, $\sqrt{c^2 t^2 - x^2}$, with the properties of a metric.